Localization of thermal packets and metastable states in the Sinai model

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We consider the Sinai model describing a particle diffusing in a one-dimensional random force field. As shown by Golosov, this model exhibits a strong localization phenomenon for the thermal packet: all thermal trajectories starting from the same initial condition in the same sample remain within a finite distance of each other even in the limit of infinite time. More precisely, he has proved that the disorder average $P_t(y)$ of the distribution of the relative distance $y=x(t)-m(t)$ with respect to the (disorder-dependent) most probable position $m(t)$, converges in the limit $t\to\infty$, towards a distribution $P_G(y)$ defined as a functional of two independent Bessel processes. In this paper, we revisit this question of the localization of the thermal packet. We first generalize the result of Golosov by computing explicitly the joint distribution $P_\infty(y,u)$ of relative position $y=x(t)-m(t)$ and relative energy $u=U(x(t))-U(m(t))$ for the thermal packet. Next, we compute the localization parameters Y_k, representing the disorder-averaged probabilities that k particles of the thermal packet are at the same place in the infinite-time limit, and the correlation function $C(l)$ representing the disorder-averaged probability density that two particles of the thermal packet are at a distance l from each other. We, moreover, prove that our results for Y_k and $C(l)$ exactly coincide with the thermodynamic limit $L\to\infty$ of the analog quantities computed for independent particles at equilibrium in a finite sample of length L. So even if the Sinai dynamics on the infinite line is always out-of-equilibrium since it consists in jumps in deeper and deeper wells, the particles of the same thermal packet can nevertheless be considered asymptotically as if they were at thermal equilibrium in a Brownian potential. Finally, we discuss the properties of the finite-time metastable states that are responsible for the localization phenomenon and compare with the general theory of metastable states in glassy systems, in particular as a test of the Edwards conjecture.

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I. INTRODUCTION

The Sinai model [1] of a particle diffusing in a one-dimensional quenched random force field is one of the simplest example of a model with quenched randomness. Its continuum version is defined by the Langevin equation

$$\frac{dx(t)}{dt}=-U'[x(t)]+\eta(t) \tag{1}$$

where $\eta(t)$ is the thermal noise, with correlation $\langle\eta(t)\eta(t')\rangle=2T\delta(t-t')$, and where the random potential $U(x)$ is a Brownian motion presenting the correlations

$$\overline{[U(x)-U(x')]^2}=2\sigma|x-x'| \tag{2}$$

As a result, the Sinai diffusion exhibits a nontrivial ultraslow logarithmic behavior, the walker typically moving as $x\sim(\ln t)^2$. Although this model has been much studied [1,3–5], the known analytical results mainly concern the rescaled variable $X=\sigma x/(T^2\ln^2 t)$, and its distribution over the disorder realizations, known as the Kesten distribution. However, another important issue concerns the thermal distribution of the position in a given sample.

Golosov [2] has discovered the important phenomenon of localization in the sense that all thermal trajectories starting from the same initial condition remain within a finite distance of each other even in the limit of infinite time. More precisely, he has proved that there exists a process $m(t)$, independent of the thermal noise η, such that the distribution $P_t(y)$ of the relative distance $y=x-m(t)$ averaged over the realizations of the disorder converges in the limit $t\to\infty$ towards a probability distribution $P_G(y)$ defined as the following functional

$$P_G(y)=\left\langle\left\langle\frac{e^{-r(|y|)}}{\int_0^\infty dt e^{-r(t)}+\int_0^\infty dt e^{-\rho(t)}}\right\rangle\right\rangle_{\{r,\rho\}} \tag{3}$$

where $\langle\langle\cdots\rangle\rangle$ denotes the average over the two independent Bessel processes (i.e., the radial parts of free Brownian motions in three dimension) $r(t)$ and $\rho(t)$ starting at $r(t=0)=0=\rho(t=0)$. (The explicit computation of this functional is done in Appendix D of the present paper.)

However, the existence of this limit distribution does not imply that the moments of the random variable y remain finite in the limit of infinite time. And indeed, all the integer moments of the relative distance $[x-\langle x(t)\rangle]$ to the thermally averaged position $\langle x(t)\rangle$ diverge in the infinite-time limit, with the following leading divergence [7,8]:

$$\overline{\langle[x-\langle x(t)\rangle]^n\rangle}\sim\frac{T}{\sigma^n}(T\ln t)^{2n-1}, \tag{4}$$

where $\langle\cdots\rangle$ denotes the thermal average over $\{\eta(t)\}$ and where $\overline{\cdots}$ denotes the disorder average over the random potential $\{U(x)\}$. This happens because the decay of the distribution of $z=x-\langle x(t)\rangle$ is algebraic at large distance as $1/|z|^{3/2}$ [8,7]. For $n=2$, the behavior (4) has been measured numerically in Ref. [9].

Other quantities characterizing the localization of the thermal packet are the localization parameters $Y_k(t)$ representing the disorder averages of the probabilities that k independent particles in the same sample starting from the same initial condition are at the same place at time t. In a given environment $U(x)$ and for a given initial condition x_0, the probability distribution over the thermal noise

$$P(x,t|x_0,0)\equiv\langle\delta(x-x_{\{\eta,U,x_0\}}(t))\rangle \tag{5}$$

[where $x_{\{\eta,U,x_0\}}(t)$ is the solution of the Langevin equation (1)] satisfies the Fokker-Planck equation

$$\partial_t P(x,t|x_0,0)=-H_{FP}P((x,t|x_0,0)), \tag{6}$$

$$H_{FP}=-\partial_x(T\partial_x+U'(x)) \tag{7}$$

and the initial condition $P(x,t\to 0|x_0,0)\to\delta(x-x_0)$. So the localization parameters read

$$Y_k(t)=\int_{-\infty}^{+\infty}dx\overline{[P(x,t|x_0,0)]^k}. \tag{8}$$

In Ref. [9], the parameter $Y_2(t)$ has been measured numerically for a version of the discrete Sinai model in a semi-infinite geometry with binary distribution of the random forces. This simulation shows that $Y_2(t)$ converges at long time towards a finite value $Y_2(\infty)$, which decays as T increases (since the temperature broadens the distribution of the thermal packet).

A generalization of $Y_2(t)$ is the correlation function $C(l,t)$ representing the disorder average of the probability that two independent particles in the same sample starting from the same initial condition are at a distance l from each other at time t,

$$C(l,t)=2\int_{-\infty}^{+\infty}dx\overline{[P(x,t|x_0,0)P(x+l,t|x_0,0)]}, \tag{9}$$

which is normalized to $\int_0^{+\infty}dl C(l,t)=1$.

Another question related to the distribution of the thermal packet is the dynamics of a given particle between two times t_w and $(t_w+\tau)$ in the "quasiequilibrium regime" $t_w\to\infty$ with finite τ [8]. It was conjectured and checked numerically in Ref. [8] that the disorder-averaged probability, $Q(z,\tau)=\lim_{t_w\to\infty}Q(z,t_w+\tau,t_w)$, of the relative displacement, $z=x(t)-x(t_w)$, for the Sinai model on the infinite line was the same as $Q_{eq}(z,\tau)$ obtained as the thermodynamic limit $L\to\infty$ of $Q_L(z,\tau)$ characterizing the equilibrium dynamics in a finite sample of length L. In particular, in the large τ limit, one should recover the statics with two independent particles at Boltzmann equilibrium [8]. If these assumptions are true, this means that for the particles of the thermal packets, we should also have a correspondence with the Boltzmann distribution in a Brownian potential on finite sample of length L,

$$p_L^{eq}(x)=\frac{e^{-\beta U(x)}}{\int_0^L dy e^{-\beta U(y)}}. \tag{10}$$

More precisely, it is interesting to compare $Y_k(\infty)$ [Eq. (8)] and $C(l,\infty)$ [Eq. (9)] with the thermodynamic limit of their statics counterparts

$$Y_k^{eq}=\lim_{L\to\infty}\int_0^L dx\overline{[p_L^{eq}(x)]^k}, \tag{11}$$

$$C^{eq}(l)=\lim_{L\to\infty}\int_0^L dx\int_0^L dy\overline{p_L^{eq}(x)p_L^{eq}(y)}. \tag{12}$$

Some statistical properties of the Boltzmann distribution (10) have already been studied in Refs. [11–13].

In this paper, we reconsider this question of the localization of the thermal packet from the point of view of the real-space renormalization group (RG) analysis detailed in Ref. [7]. Within this renormalization picture, at time t, any particle starting from an initial condition belonging to a renormalized valley will be typically at time t around the minimum $m(t)$ of this renormalized valley. To study the distribution of a thermal packet, a first step is to consider that the particles of the packet are at Boltzmann equilibrium within the renormalized valley they belong to at time t. This is only an approximation at finite time, since there are also additional out-of-equilibrium situations for the thermal packet [7]. However, in the limit of infinite time, these out-of-equilibrium situations have vanishing probability [7], and the joint distribution $P_\infty(y,u)$ of relative position $y=x(t)-m(t)$ and relative energy $u=U(x(t))-U(m(t))$ corresponds to an average of Boltzmann distribution over infinitely deep Brownian valleys. We will compute explicitly $P_\infty(y,u)$ by a path-integral method. We use the same method to compute the $Y_k(\infty)$ parameters (8) and the correlation function $C(l)$ [Eq. (9)]. On the other hand, we compute Y_k^{eq} and $C^{eq}(l)$ [Eq. (12)] and find that they indeed coincide with $Y_k(\infty)$ and $C(l,\infty)$. This shows that the ensemble of infinitely deep valleys gives the same results for the quantities mentioned above as the thermodynamic limit of the ensemble of finite-size valleys, so that quasiequilibrium in Sinai diffusion and equilibrium in a Brownian potential are equivalent.

This approach to the localization phenomenon allows us to study in details the disorder-dependent structure of low-energy eigenstates of the Fokker-Planck operator. Our results are consistent with the features discussed in Appendix B of Ref. [14] for the related model of one-dimensional random-hopping Hamiltonian for fermions.

Finally, it is instructive to recast Sinai diffusion into the general theory of glassy systems. Indeed, in the studies on slowly relaxing systems such as glasses, granular media or disordered spin models, it is natural to separate the dynamics into two parts: there are "fast" degrees of freedom that rapidly reach local quasiequilibrium plus a slow nonequilibrium part. At a given long-time t, the fast motion covers a region of phase space which can be defined as a metastable state associated with time t [15]. In Sinai diffusion, this picture directly applies: the metastable states are the renormalized valleys, within which there is a local Boltzmann equilibrium. Moreover, we obtain that the metastable states satisfy all the properties of the construction [16] as summarized in Ref. [15]. In glassy systems, the Edwards ergodicity conjecture [17] consisting in computing dynamic quantities by taking flat averages over metastable states has given rise to a lot of recent studies [15,18–21]. Since Edwards conjecture is usually based on the assumption that all the basins of attraction of the various metastable states have the same size [15], it is of course a very strong hypothesis that cannot be true in general but only for special systems with special dynamics [18,21]. In Sinai diffusion with uniform initial condition, the size of the basin of attraction of a metastable state is given by the spatial length of a renormalized valley: it is thus a random variable whose distribution is exactly known. As a consequence, Edwards conjecture cannot be true in general. Nevertheless, within the real-space RG (RSRG) approach [7], all one-time quantities are effectively computed by averages over all metastable states, but with a measure that is not flat, but depends on the quantities and on the properties of the basins of attraction. So the RSRG approach of the Sinai model represents the simplest example where the dynamical study of a glassy system can be faithfully replaced by an average over a set of well-specified metastable states, with a well-defined measure. However, for special quantities, Edwards conjecture can be recovered. For instance, in this paper, we compute the probability distribution $P_\infty(y,u)$ of the thermal packet, the localization parameters $Y_k(\infty)$ [Eq. (8)] and the correlation function $C(l,\infty)$ [Eq. (9)] as flat averages over infinitely deep wells. This is because in the infinitely deep valleys, the statistics of the lower-part of the Brownian valley is the same for all metastable states.

The paper is organized as follows. In Sec. II, we explain within the RSRG picture why the distribution of the thermal packet is asymptotically given by an average of Boltzmann distribution over infinitely deep Brownian valleys. In Sec. III, we use a probabilistic path-integral method to compute explicit expressions for the joint probability distribution $P_\infty(y,u)$. We use the same method to compute the probability distribution of the partition function of an infinitely deep valley (Sec. IV), the localization parameters $Y_k(\infty)$ (Sec. V), and the correlation function $C(l,\infty)$ (Sec. VI). In Sec. VII, we consider equilibrium functions in finite samples and compute Y_k^{eq} and $C^{eq}(l)$ [Eq. (12)] that are found to coincide with $Y_k(\infty)$ and $C(l,\infty)$. In Sec. VIII, we derive explicit expressions for the eigenfunctions of the Fokker-Planck operator. In Sec. IX, we discuss the properties of metastable stables and compare with the general theory of metastable states in glassy systems. Finally, the Appendices A, B, and C contain technical details used in the text, whereas Appendix D shows that our result for $P_\infty(y,u)$ after integration over u coincides with the explicit computation of the Golosov functional (3).

II. REAL-SPACE RENORMALIZATION GROUP FOR THE SINAI DIFFUSION

In this section, we briefly recall the principles of the real-space renormalization group approach to Sinai diffusion [7] with special emphasis on the successive levels of approximations.

A. Effective dynamics at large times

Recently we have proposed an approach, based on RSRG method, which allows us to obtain many exact results for the Sinai model [6,7]. The way to implement the RSRG is very direct: one decimates iteratively the *smallest-energy barrier* in the system stopping when the time to surmount the smallest remaining barrier is of order the time scale of interest. Despite its approximate character, the RSRG yields for many quantities asymptotically exact results, because the iterated distribution of barriers grows infinitely wide. Indeed, the distribution of the rescaled barrier height $\eta=(F-\Gamma)/\Gamma$ converges towards the fixed point

$$P^*(\eta)=\theta(\eta)e^{-\eta}, \tag{13}$$

where

$$\Gamma=T\ln t \tag{14}$$

is the renormalization scale associated with the time t.

Within this renormalization picture, at time t, any particle starting from an initial condition belonging to a renormalized valley (F_1,F_2) will be typically at time t around the minimum $m(t)$ of the valley. This simple approximation, called "effective dynamics" in Ref. [7], is sufficient to obtain *exact* expressions for many quantities, such as for instance the Kesten distribution of the rescaled variable $X=\sigma x(t)/(T^2\ln^2 t)$.

However, for other quantities, we have already obtained in Ref. [7] that there are differences between the effective dynamics and the real dynamics. For instance, persistence properties of the thermal average $\langle x(t)\rangle$ are well described by persistence properties of the effective dynamics consisting in jumping between valley bottoms but are very different from the persistence properties of a single walker [7].

B. Boltzmann equilibrium within renormalized valleys

To study the distribution of a thermal packet, we clearly need to go beyond the effective dynamics. A first step is to consider that the particles of the packet are at equilibrium within the renormalized valley they belong to at time t. More explicitly, this approximation which assumes that the walkers are at Gibbs equilibrium separately in each renormalized valleys at scale Γ, can be written as

$$P(xt|x_0 0)\simeq\sum_{V_\Gamma}\frac{1}{Z_{V_\Gamma}}e^{-\beta U(x)}\theta_{V_\Gamma}(x)\theta_{V_\Gamma}(x_0), \tag{15}$$

where the sum is over all the renormalized valleys V_Γ that are present in the system at the renormalization scale $\Gamma=T\ln t$, and where $\theta_V(x)$ is the characteristic function of the valley V, i.e., $\theta_V(x)=1$ if x belongs to the valley and $\theta_V(x)=0$ otherwise. $Z_V=\int_V dx e^{-\beta U(x)}$ represents the Boltzmann normalization over the valley V.

This Boltzmann equilibrium is clearly an excellent approximation within the lower part of the valley, i.e., for the points that are at a finite potential above the minimum of the valley, which have had plenty of time to equilibrate. However, it breaks down further away in the higher part of the valley, for the points that are at a potential of order Γ above the minimum of the valley, since these points need a time of order $e^{\beta\Gamma}\sim t$ to equilibrate. However, since these points have a weight of order $e^{-\beta\Gamma}$ in the formula (15), they do not play any role for the observables computed in this paper in the limit $\Gamma\to\infty$.

More importantly, the approximation (15) breaks down whenever out-of-equilibrium situations occur for the thermal packet as we now explain.

C. Out-of-equilibrium situations for a thermal packet

In our previous work [7], we have already described rare events where deviations from the effective dynamics show up. The most important rare events are of order $1/\Gamma$ and are of three types as shown in Fig. 7 of Ref. 7. In the events of type (a), there are two nearly degenerate minima at thermal equilibrium separated by a barrier $\Gamma_0<\Gamma$. These events (a) are thus taken into account well by the Boltzmann equilibrium in each renormalized valley described in the preceding section. The rare events (b) where two tops are nearly degenerate are on the contrary completely out of equilibrium, since the thermal packet will be split in two valleys that are not at equilibrium with each other. Finally, the events (c) where the valley is being decimated are also out-of-equilibrium events, since the two renormalized valleys at Γ cannot be considered to be at thermal equilibrium. All other rare events are of higher order, for instance, the probability that the initial point is near a top, which will also produce an out-of-equilibrium splitting of the thermal packet, is of order $1/\Gamma^2$.

D. Conclusion

As a conclusion, the expression (15) is an excellent approximation for the thermal packets that are not in out-of-equilibrium situations, and within the lower part of the renormalized valleys, i.e., for the points that are at a finite potential above the minima of the valleys. This approximation breaks down for the higher parts of the valleys, i.e., for the points that are at a potential of order Γ above the minima of the valleys, and whenever the thermal packet happens to be in an out-of-equilibrium situation like the events (b) and (c) described above which appear with probability $1/\Gamma$. For a detailed study of systematic corrections to this approximation, and further results, we refer the reader to Fisher [23].

From now on, in this paper, we will restrict our attention to the approximation (15) that becomes asymptotically exact in the limit $\Gamma\to\infty$. Indeed, in this limit, only the points that are at a finite potential above the minima of the valleys keep a finite weight and all the out-of-equilibrium situations have a vanishing probability in the limit $\Gamma\to\infty$. As a consequence, the Boltzmann distribution over an infinite-deep valley exactly describes the asymptotic dispersion of a thermal packet. More explicitly, the disorder average $P_\infty(y,u)$ of the joint probability distribution of the relative position $y=x(t)-m(t)$ and the relative energy $u=U(x(t))-U(m(t))$ with respect to the minimum $[m(t),U(m(t)]$ of the valley is given by

$$P_\infty(y,u)=\left\langle \frac{e^{-\beta u}\delta(u-U_1(|y|))}{\int_0^\infty dx e^{-\beta U_1(x)}+\int_0^\infty dx e^{-\beta U_2(x)}}\right\rangle_{\{U_1,U_2\}}, \tag{16}$$

where the average $\langle\cdots\rangle$ is over two independent Brownian trajectories $U_1(x)$ and $U_2(x)$ *forming an infinitely deep well*, i.e., a renormalized valley in the limit $\Gamma\to\infty$. We note here that the minimum $m(t)$ of the valley represents the most probable position in each sample (i.e., it is the point where the probability is the biggest), but not the thermally averaged position $\langle x(t)\rangle$.

Using, the same notations, we obtain the infinite-time limit of the localization parameters $Y_k(\infty)$ [Eq. (8)]

$$Y_k(\infty)=\int_{-\infty}^{+\infty}dy\left\langle\left(\frac{e^{-\beta U_1(|y|)}}{\int_0^{+\infty}dx e^{-\beta U_1(x)}+\int_0^{+\infty}dx e^{-\beta U_2(x)}}\right)^k\right\rangle_{\{U_1,U_2\}}, \tag{17}$$

and of the correlation function (9)

$$C(l,\infty)=4\int_0^\infty dy\overline{[P_\infty(y)P_\infty(y+l)]}+2\int_0^l dy\overline{[P_\infty(l-y)P_\infty(y)]}=4\int_0^\infty dy\left\langle\frac{e^{-\beta U_1(y)-\beta U_1(y+l)}}{\left(\int_0^\infty dx e^{-\beta U_1(x)}+\int_0^\infty dx e^{-\beta U_2(x)}\right)^2}\right\rangle$$
$$+2\int_0^l dy\left\langle\frac{e^{-\beta U_1(y)-\beta U_2(l-y)}}{\left(\int_0^{+\infty}dx e^{-\beta U_1(x)}+\int_0^{+\infty}dx e^{-\beta U_2(x)}\right)^2}\right\rangle. \tag{18}$$

We now turn to the explicit computation of these expressions.

III. PROBABILITY DISTRIBUTION FOR THE THERMAL PACKET

A. Expression of $P_\infty(y,u)$ in terms of path integrals

To define more precisely the average $\langle \cdots \rangle$ [Eq. (16)] over two independent Brownian trajectories $U_1(x)$ and $U_2(x)$ *forming an infinitely deep well*, let $\mathcal{V}_\Gamma$ be the set of Brownian paths $\{U(x\geqslant 0)\}$ starting at $U(0)=0^+$ in the presence of absorbing boundaries at 0 and Γ, and that are conditioned to finish at $U=\Gamma$ and not at $U=0$. The formula (16) is thus defined as

$$P_\infty(y,u)=\lim_{\Gamma\to\infty}\left\langle \frac{e^{-\beta u}\delta(u-U_1(|y|))}{\int_0^{l_\Gamma^{(1)}}dxe^{-\beta U_1(x)}+\int_0^{l_\Gamma^{(2)}}dxe^{-\beta U_2(x)}}\right\rangle, \tag{19}$$

where $U_1(x)$ and $U_2(x)$ are two independent Brownian trajectories belonging to $\mathcal{V}_\Gamma$ and where $l_\Gamma^{(1)}$ and $l_\Gamma^{(2)}$ are the random times where $U_1(x)$ and $U_2(x)$, respectively, first hit $x=\Gamma$ where they are killed. Since the expression is symmetric in $y\to -y$, we will assume $y>0$ from now on.

To separate the averages over U_1 and U_2, it is convenient to exponentiate the denominator to get

$$P_\infty(y,u)=\lim_{\Gamma\to\infty}R_\Gamma(q)S_\Gamma(y,u,q), \tag{20}$$

where

$$R_\Gamma(q)\equiv\left\langle \exp\left[-q\int_0^{l_\Gamma^{(2)}}dx\, e^{-\beta U_2(x)}\right]\right\rangle_{\{U_2\}} \tag{21}$$

and

$$S_\Gamma(y,u,q)\equiv\left\langle e^{-\beta u}\delta(u-U_1(|y|)) \times\exp\left[-q\int_0^{l_\Gamma^{(1)}}dx\, e^{-\beta U_1(x)}\right]\right\rangle_{\{U_1\}}. \tag{22}$$

We now define the path integral

$$F_{[0,\Gamma]}(u,l|u_0)\equiv\int_{U(0)=u_0}^{U(l)=u}\mathcal{D}U(x)\exp\left[-\frac{1}{4\sigma}\int_0^l dx\left(\frac{dU}{dx}\right)^2 -q\int_0^l dxe^{-\beta U(x)}\right]\Theta_{[0,\Gamma]}\{U\}, \tag{23}$$

where $\Theta_{[0,\Gamma]}\{U(x)\}$ means that there are absorbing boundaries at $U=0$ and at $U=\Gamma$. The explicit computation of this path integral is done in Appendix B and yields the final result [Eq. (B18)].

To compute the quantity (21), we need to consider the path integral (23) going from the initial potential $u_0=\epsilon$ to the final potential $u=\Gamma-\epsilon$ in the limit $\epsilon\to 0$ and to sum over the random time l representing the random time $l_\Gamma^{(2)}$ where $U_2(x)$ first hit $x=\Gamma$ where it is killed. So we have

$$R_\Gamma(q)=\mathcal{N}(\Gamma)\int_0^{+\infty}dl\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}F_{[0,\Gamma]}(\Gamma-\epsilon,l|\epsilon) \tag{24}$$

up to a normalization $\mathcal{N}(\Gamma)$ that ensures $R_\Gamma(q=0)=1$. The result for $R_\Gamma(q)$ is given in Eq. (B25) of Appendix B that yields in the limit $\Gamma\to\infty$,

$$R_\infty(q)=\frac{1}{I_0\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}\right)}. \tag{25}$$

Similarly, to compute Eq. (22), we need to compose two path integrals of type (23), the first one going from the initial potential $u_0=\epsilon$ to the final potential u in a time y, and the second one going from the initial potential u to the final potential $\Gamma-\epsilon$ in a time l representing the difference $(l_\Gamma^{(1)}-y)$ that we have to sum over, so that we have

$$S_\Gamma(y,u,q)=\mathcal{N}(\Gamma)e^{-\beta u}\int_0^{+\infty}dl\lim_{\epsilon\to 0}\frac{1}{\epsilon^2} \times F_{[0,\Gamma]}(\Gamma-\epsilon,l|u)F_{[0,\Gamma]}(u,y|\epsilon). \tag{26}$$

The Laplace transform with respect to y of this expression is given in Eq. (B30) of Appendix B which yields in the limit $\Gamma\to\infty$

$$\hat{S}_\infty(p,u,q)\equiv\int_0^{+\infty}dye^{-py}S_\Gamma(y,u,q) =\frac{2}{\beta\sigma}e^{-\beta u}\frac{I_\nu(se^{-\beta u/2})}{I_\nu(s)}\left(K_0(se^{-\beta u/2}) -\frac{K_0(s)}{I_0(s)}I_0(se^{-\beta u/2})\right), \tag{27}$$

where

$$s=\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}, \tag{28}$$

$$\nu=\frac{2}{\beta}\sqrt{\frac{p}{\sigma}}. \tag{29}$$

B. Final result for the joint probability distribution $P_\infty(y,u)$

Using the results (25) and (27), the Laplace transform of the probability distribution $P_\infty(y,u)$ [Eq. (20)] may now be expressed as

$$\hat{P}_\infty(p,u)\equiv\int_0^{+\infty}dye^{-py}P_\infty(y,u) \tag{30}$$

$$
\begin{aligned}
&= \int_0^\infty dq R_\infty(q) \hat{S}_\infty(p,u,q) \\
&= \beta e^{-\beta u} \int_0^\infty dss \\
&\quad \times \frac{I_\nu(s e^{-\frac{\beta u}{2}})}{I_0(s) I_\nu(s)} \left(K_0(s e^{-\beta u/2}) \right. \\
&\quad \left. - \frac{K_0(s)}{I_0(s)} I_0(s e^{-\beta u/2}) \right). \qquad (31)
\end{aligned}
$$

In particular, the distribution of the energy u alone reads (taking into account the two sides $y>0$ and $y<0$)

$$
\begin{aligned}
P_\infty(u) &= 2 \hat{P}_\infty(p=0,u) \\
&= 2 \beta e^{-\beta u} \int_0^\infty dss \frac{I_0(s e^{-\beta u/2})}{I_0^2(s)} \left(K_0(s e^{-\beta u/2}) \right. \\
&\quad \left. - \frac{K_0(s)}{I_0(s)} I_0(s e^{-\beta u/2}) \right), \qquad (32)
\end{aligned}
$$

whereas the distribution of the position y alone has for Laplace transform

$$
\begin{aligned}
\hat{P}_\infty(p) &\equiv \int_0^{+\infty} du \hat{P}_\infty(p,u) \\
&= 2 \int_0^\infty \frac{ds}{s I_0(s) I_\nu(s)} \int_0^s dzz I_\nu(z) \\
&\quad \times \left(K_0(z) - \frac{K_0(s)}{I_0(s)} I_0(z) \right). \qquad (33)
\end{aligned}
$$

The Laplace parameter p is present only through the index $\nu = (2/\beta)\sqrt{p/\sigma}$ of the Bessel function I_ν.

The normalization to $\hat{P}_\infty(p=0) = 1/2$ for the half space can be checked using Eqs. (A7) and (A10). Using Eq. (A15), we may expand in ν as follows:

$$
\hat{P}_\infty(p) = \frac{1}{2} - \frac{2}{3} \nu \int_0^\infty dzz \frac{K_0^3(z)}{I_0(z)} + O(\nu^2) \qquad (34)
$$

$$
= \frac{1}{2} - \frac{4}{3\beta} \sqrt{\frac{p}{\sigma}} \int_0^\infty dzz \frac{K_0^3(z)}{I_0(z)} + O(p). \qquad (35)
$$

This shows that the probability distribution $P_\infty(y)$ exhibits the power-law decay

$$
P_\infty(y) \underset{|y|\to\infty}{\propto} \frac{1}{|y|^{3/2}} \left(\frac{2T}{3\sqrt{\pi\sigma}} \int_0^\infty dzz \frac{K_0^3(z)}{I_0(z)} \right), \qquad (36)
$$

making all the integer moments diverging in the limit $t \to \infty$. The exponent 3/2 is of course related to the probability of return to the origin of a random walk [8,7]. Indeed, the Boltzmann distribution in a renormalized valley typically decays at large distance as $e^{-\beta\sqrt{\sigma y}}$. However, in rare configurations where the random potential $U(y)$ happens to be near the origin $U(0)$ at y, which happens with probability $1/y^{3/2}$, then the Boltzmann distribution has a weight of order 1 at y. These rare configurations entirely dominate the disorder average for large y and are responsible for the power-law decay [8,7].

IV. DISTRIBUTION OF THE PARTITION FUNCTION OF AN INFINITELY DEEP VALLEY

The partition function of the valley can be decomposed as the sum over two independent half valleys

$$
Z_\infty \equiv \int_0^{+\infty} dx e^{-\beta U_1(x)} + \int_0^\infty dx e^{-\beta U_2(x)}. \qquad (37)
$$

Using the result (25) for $R_\Gamma(q)$ [Eq. (21)], we obtain that its probability distribution has for Laplace transform

$$
\int_0^{+\infty} dZ e^{-qZ} \mathcal{P}_\infty(Z) = R_\Gamma^2(q) = \frac{1}{I_0^2\left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} \right)}. \qquad (38)
$$

It is convenient to introduce the rescaled partition function

$$
z \equiv \frac{Z_\infty}{l_T}, \qquad (39)
$$

where

$$
l_T = \frac{4T^2}{\sigma} \qquad (40)
$$

represents the thermal length associated with the typical scale of the extension of the Boltzmann distribution $e^{-\beta U(x)}$ in a Brownian well $U(x) \sim \sigma\sqrt{x}$.

The probability distribution $p(z)$ of the dimensionless partition function z has for Laplace transform

$$
\int_0^{+\infty} dz e^{-qz} p(z) = \frac{1}{I_0^2(\sqrt{q})}. \qquad (41)
$$

The series expansion (A3) shows that all the positive moments are finite. The leading behavior at large z is indeed given via Laplace inversion by the first pole $q_1 = -s_1^2$ on the negative real axis, where s_1 is the first zero of the Bessel function $J_0(s_1) = 0$,

$$
p(z) \underset{z\to\infty}{\sim} z e^{-s_1^2 z}. \qquad (42)
$$

The behavior of Eq. (41) at large q,

$$
\int_0^{+\infty} dz e^{-qz} p(z) \underset{q\to\infty}{\simeq} 2\pi\sqrt{q} e^{-2\sqrt{q}}, \qquad (43)
$$

leads to the following essential singularity at small z:

$$p(z) \underset{z\to 0}{\sim} \frac{1}{z^{5/2}} e^{-1/z}. \tag{44}$$

V. LOCALIZATION PARAMETERS

To compute the localization parameters $Y_k(\infty)$ [Eq. (17)] we proceed along the same lines,

$$Y_k(\infty)=2\int_0^{+\infty} dy \frac{1}{\Gamma(k)} \int_0^{+\infty} dq q^{k-1} R_\infty(q) \times \lim_{\Gamma\to\infty} \left\langle e^{-k\beta U_1(y)} \exp\left[-q\int_0^{l_\Gamma^{(1)}} dx e^{-\beta U_1(x)}\right]\right\rangle_{\{U_1\}} \tag{45}$$

$$=\frac{2}{\Gamma(k)}\int_0^{+\infty} dq q^{k-1} R_\infty(q) \lim_{\Gamma\to\infty} \mathcal{N}(\Gamma) \times \int_0^{\Gamma} du e^{-k\beta u} \lim_{\epsilon\to 0}\frac{1}{\epsilon^2} \hat{F}_{[0,\Gamma]}(\Gamma-\epsilon,l|u)\hat{F}_{[0,\Gamma]}(u,0|\epsilon). \tag{46}$$

Using again the result (B18) for $\hat{F}_{[0,\Gamma]}$ we finally get

$$Y_k(\infty)=\frac{2}{\Gamma(k)}\left(\frac{\beta^2}{4}\sigma\right)^{k-1}\int_0^\infty dz z^{2k-1}K_0^2(z) = \frac{\sqrt{\pi}\Gamma^2(k)}{2\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{\beta^2}{4}\sigma\right)^{k-1} = \frac{\Gamma^3(k)}{\Gamma(2k)}(\beta^2\sigma)^{k-1}, \tag{47}$$

where we have used again Eqs. (A7) and (A10).

The increase of Y_k at large k is a consequence of the average over the disorder, and can be understood by considering the averaged probability $\tilde{Y}_k$ to have k particles at the minimum of the valley (instead of at the same place but anywhere in the valley for Y_k), which is exactly given by the negative moment of order k of the partition function Z_∞ [Eq. (37)]

$$\tilde{Y}_k=\left\langle \frac{1}{Z^k}\right\rangle = \int_0^{+\infty} dZ P_\infty(Z)\frac{1}{Z^k} = \frac{2}{\Gamma(k)}\left(\frac{1}{l_T}\right)^k \int_0^{+\infty} ds \frac{s^{2k-1}}{I_0^2(s)}. \tag{48}$$

For large k, the dominant behavior comes from the the small z behavior (44) of the probability distribution, which yields

$$\tilde{Y}_k \underset{k\to\infty}{\sim} \left(\frac{1}{l_T}\right)^k \Gamma\left(k+\frac{1}{2}\right). \tag{49}$$

This shows that the behavior at large k of the average Y_k is dominated by very steep valleys having a small partition function z.

Here we need to make some comments about the relation with the discrete Sinai model with lattice constant a. For most quantities, the results obtained for the continuum version correspond to the universal limit where the lattice constant a is very small as compared to the thermal length $l_T \sim 1/(\sigma\beta^2)$ [Eq. (40)] representing the typical extension of the Boltzmann distribution in a Brownian well. For instance, this is the case for the probability distribution $P_\infty(y,u)$ of the thermal packet and for the correlation function $C(l,\infty)$. For the localization parameters Y_k, however, our result indicates that the dimensionless y_k parameters of discrete models will behave as

$$y_k^{discrete}(\infty)=\frac{\sqrt{\pi}\Gamma^2(k)}{2\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{a}{l_T}\right)^{k-1}, \tag{50}$$

when k is fixed, in the limit where (a/l_T) is small. But for fixed (a/l_T), there is a maximal value k_{max} beyond which the result above does not apply anymore and is replaced by a nonuniversal behavior. Indeed, the discrete y_k are by definition smaller than 1. And as explained above, the large k behavior of Eq. (47) is related to very steep valleys having a small partition function z, i.e., involving a small number of sites in discrete models, so that all details of the model will be important to determine the large k behavior of $y_k^{discrete}(\infty)$.

Note also that in the opposite regime where the lattice spacing a is not negligible with the thermal length l_T (i.e., $a\sim l_T$ or $a>l_T$), there is only a small number of sites that are really important around the minimum of the valley so that the discreteness and details of the model will again be very important. For instance, the behaviors of the localization parameters $Y_k(\infty)$ at zero temperature are highly nonuniversal and depend on many details: in Ref. [9], the binary distribution of the random forces induces a lot of minima degeneracies separated by barrier of two bonds that can always be passed even in the the limit of zero-temperature (because the particle is not allowed to remain on the same site between t and $t+1$). Assuming that all degenerate minima have the same weight, the value of $Y_2(\infty)$ at $T=0$ is found to be $(\ln 2)/2$ [9], instead of the 1 one would expect if there were no residual fluctuations at $T=0$ around a single minimum.

VI. CORRELATION FUNCTION

To compute the correlation function $C(l,\infty)$ [Eq. (18)], we decompose it into

$$C(l,\infty)=\lim_{\Gamma\to\infty}2\int_0^{\infty}dqq\int_0^{l}dy\left\langle e^{-\beta U_1(y)}\exp\left[-q\int_0^{l_\Gamma^{(1)}}dxe^{-\beta U_1(x)}\right]\right\rangle_{\{U_1\}}\left\langle e^{-\beta U_2(l-y)}\exp\left[-q\int_0^{l_\Gamma^{(2)}}dxe^{-\beta U_2(x)}\right]\right\rangle_{\{U_2\}}$$
$$+\lim_{\Gamma\to\infty}4\int_0^{\infty}dqq\left\langle\exp\left[-q\int_0^{l_\Gamma^{(2)}}dxe^{-\beta U_2(x)}\right]\right\rangle_{\{U_2\}}\int_0^{\infty}dy\left\langle e^{-\beta U_1(y)-\beta U_1(y+l)}\exp\left[-q\int_0^{l_\Gamma^{(1)}}dxe^{-\beta U_1(x)}\right]\right\rangle_{\{U_1\}}, \tag{51}$$

which yields in Laplace transform with respect to l

$$\hat{C}(p,\infty)\equiv\int_0^{+\infty}dle^{-pl}C(l,\infty)=2\int_0^{\infty}dqq\lim_{\Gamma\to\infty}\left(\int_0^{\Gamma}du\int_0^{+\infty}dye^{-py}S_\Gamma(y,u,q)\right)^2$$
$$+4\int_0^{\infty}dqqR_\infty(q)\lim_{\Gamma\to\infty}\mathcal{N}(\Gamma)\int_0^{\Gamma}du_1e^{-\beta u_1}\int_0^{\Gamma}du_2e^{-\beta u_2}\hat{F}_{[0,\Gamma]}(u_2,p|u_1)\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\hat{F}_{[0,\Gamma]}(\Gamma-\epsilon,0|u_2)\hat{F}_{[0,\Gamma]}(u_1,0|\epsilon). \tag{52}$$

Using the previous result (27) and the expression (B18) for $\hat{F}_{[0,\Gamma]}$, we finally get after simplifications

$$\hat{C}(p,\infty)=8\int_0^{+\infty}dz_1z_1I_\nu(z_1)K_0(z_1)\times\int_{z_1}^{+\infty}dz_2z_2K_\nu(z_2)K_0(z_2), \tag{53}$$

where again p only appears in the index ν [Eq. (29)] of Bessel functions.

Expansion in p yields [Eq. (A15)]

$$\hat{C}(p,\infty)=1-\frac{2}{\beta}\sqrt{\frac{p}{\sigma}}+O(p). \tag{54}$$

This small-p behavior shows that $C(l,\infty)$ presents at large distance the same power-law decay with exponent (3/2) as the probability distribution $P_\infty(y)$ [Eq. (36)]

$$C(l,\infty)\underset{l\to\infty}{\propto}\frac{1}{l^{3/2}}\left(\frac{T}{\sqrt{\pi\sigma}}\right), \tag{55}$$

making again all the integer moments infinite. As explained after Eq. (36), this long-range algebraic decay of the mean correlation function comes from rare configurations of the disorder, and is very different from the decay as $e^{-\beta\sqrt{\sigma l}}$ characterizing the typical correlations. This is thus an explicit example of the important differences that exist in disordered systems between typical and mean correlations [10].

VII. COMPARISON WITH EQUILIBRIUM FUNCTIONS IN LARGE SYSTEMS

In this section, we consider the Boltzmann equilibrium (10) on a finite system of length L to see if, in the thermodynamic limit $L\to\infty$, we recover the same properties for the thermal packet as in Sinai diffusion on the infinite line. Some statistical properties of the Boltzmann distribution (10) have already been studied in Refs. [11–13], where in particular the decay of correlations was shown to be algebraic with the exponent 3/2, for the same reason as discussed above after Eq. (36). The Y_k^{eq} have already been computed in Ref. [13] for free and periodic boundary conditions, but they were found to be very different even in the thermodynamic limit $L\to\infty$, whereas it is usually expected that for physical quantities that remain finite in this limit, differences should vanish. In the following, we compute the equilibrium functions for both boundary conditions and find that they coincide with each other, [i.e., the result (17) of Ref. [13] is erroneous]. The thermodynamic limit is thus well defined and independent of the boundary conditions.

A. Y_k^{eq} parameters at equilibrium

The Y_k^{eq} parameters for free boundary conditions may be rewritten in terms of path integrals as

$$Y_k^{free}(L)=\int_0^{L}dx\overline{\frac{e^{-k\beta U(x)}}{\left(\int_0^{L}dye^{-\beta U(y)}\right)^k}} \tag{56}$$

$$=\frac{1}{\Gamma(k)}\int_0^{+\infty}dqq^{k-1}\int_0^{L}dx\ \overline{\exp[-k\beta U(x)]\exp\left(-q\int_0^{L}dye^{-\beta U(y)}\right)} \tag{57}$$

$$=\frac{1}{\Gamma(k)}\int_0^{+\infty}dq q^{k-1}\int_0^L dx\int_{-\infty}^{+\infty}du_L\int_{-\infty}^{+\infty}du e^{-k\beta u}G(u_L,L-x|u)G(u,x|0), \tag{58}$$

where the basic path integral G is

$$G(u,l|u_0)=\int_{U(0)=u_0}^{U(l)=u}\mathcal{D}U(y)\exp\left[-\frac{1}{4\sigma}\int_0^L dy\left(\frac{dU}{dy}\right)^2-q\int_0^L dy e^{-\beta U(x)}\right]. \tag{59}$$

It is analogous to the path integral (23) except that here there are no boundary conditions at $U=0$ and $U=\Gamma$, and the variable u is in $]-\infty,+\infty[$. From a technical point of view, we mention here that contrary to the previous works [11–13] that expand the path-integral (59) upon eigenstates of the the Hamiltonian $H=-d/du^2+qe^{-\beta u}$, here we have chosen to work in Laplace transform with respect to the length l to have a more compact result (C1). Using Eq. (C1), we obtain in Laplace with respect to L

$$\hat{Y}_k^{free}(\omega)\equiv\int_0^{+\infty}dL e^{-\omega L}Y_k^{free}(L) \tag{60}$$

$$=\frac{2}{\Gamma(k)}\left(\sigma\frac{\beta^2}{4}\right)^{k-2}\int_0^{+\infty}dz z^{2k-1}$$
$$\times\left[K_\mu(z)\int_0^z\frac{ds}{s}I_\mu(s)+I_\mu(z)\int_z^\infty\frac{ds}{s}K_\mu(s)\right]^2, \tag{61}$$

with

$$\mu=\frac{2}{\beta}\sqrt{\frac{\omega}{\sigma}}. \tag{62}$$

The thermodynamic limit $L\to\infty$ is obtained as

$$\lim_{L\to\infty}Y_k^{free}(L)=\lim_{\omega\to 0}[\omega\hat{Y}_k^{free}(\omega)] \tag{63}$$

$$=\frac{2}{\Gamma(k)}\left(\sigma\frac{\beta^2}{4}\right)^{k-1}\int_0^{+\infty}dz z^{2k-1}K_0^2(z) \tag{64}$$

$$=\frac{\sqrt{\pi}\Gamma^2(k)}{2\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{\beta^2}{4}\sigma\right)^{k-1}, \tag{65}$$

in agreement with Eq. (19) of Ref. [13].

We now consider periodic boundary conditions, and indicate the modifications that appear. Taking into account that the probability to have $U(L)=U(0)$ is $(1/\sqrt{4\pi\sigma L})$, we have in terms of the path integral (59),

$$Y_k^{periodic}(L)=\frac{\sqrt{4\pi\sigma L}}{\Gamma(k)}\int_0^{+\infty}dq q^{k-1}$$
$$\times\int_0^L dx\int_{-\infty}^{+\infty}du e^{-k\beta u}$$
$$\times G(0,L-x|u)G(u,x|0). \tag{66}$$

So here, it is simpler to compute the following Laplace transform

$$y_k(\omega)\equiv\int_0^{+\infty}dL e^{-\omega L}\left(\frac{Y_k^{periodic}(L)}{\sqrt{L}}\right) \tag{67}$$

$$=\frac{8\sqrt{\pi}}{\beta\sqrt{\sigma}\Gamma(k)}\left(\sigma\frac{\beta^2}{4}\right)^{k-1}\int_0^{+\infty}dz z^{2k-1}$$
$$\times\left[K_\mu^2(z)\int_0^z\frac{ds}{s}I_\mu^2(s)+I_\mu^2(z)\int_z^\infty\frac{ds}{s}K_\mu^2(s)\right]. \tag{68}$$

The thermodynamic limit $L\to\infty$ is then obtained as

$$\lim_{L\to\infty}Y_k^{periodic}(L)=\lim_{\omega\to 0}\left(\frac{\sqrt{\omega}}{\sqrt{\pi}}y_k(\omega)\right) \tag{69}$$

$$=\frac{\sqrt{\pi}\Gamma^2(k)}{2\Gamma\left(k+\frac{1}{2}\right)}\left(\frac{\beta^2}{4}\sigma\right)^{k-1}, \tag{70}$$

contrary to the erroneous result in Eq. (17) of Ref. [13]. Our result thus coincides with Eq. (65) concerning the equilibrium with free boundary conditions and with Eq. (47) concerning the localizations parameters for Sinai diffusion on the infinite-line.

B. Two-point correlation $C^{eq}(l)$ at equilibrium

Similarly, the two-point correlation for free boundary conditions may be expressed in terms of the path-integral (59)

$$C_L^{free}(l)=2\int_0^{L-l}dx\overline{[p_L^{eq}(x)p_L^{eq}(x+l)]}=2\int_0^{+\infty}dqq\int_0^{L-l}dx\int_0^{+\infty}dqq\overline{\exp[-\beta U(x)-\beta U(x+l)]\exp\left(-q\int_0^L dye^{-\beta U(x)}\right)} \tag{71}$$

$$=2\int_0^{L-l}dx\int_0^{+\infty}dqq\int_{-\infty}^{+\infty}du_1\int_{-\infty}^{+\infty}du_2e^{-\beta u_1-\beta u_2}\int_{-\infty}^{+\infty}du_LG(u_L,L-x-l|u_2)G(u_2,l|u_1)G(u_1,x|0) \tag{72}$$

so that in double Laplace with respect to l and L we get

$$\hat{C}^{free}(p,\omega)\equiv\int_0^{+\infty}dLe^{-\omega L}\int_0^L dle^{-pl}C_L^{free}(l)$$
$$=\frac{32}{\beta^2\sigma}\int_0^{+\infty}dz_1z_1I_{\nu'}(z_1)\left[K_\mu(z_1)\int_0^{z_1}\frac{ds}{s}I_\mu(s)+I_\mu(z_1)\int_{z_1}^{\infty}\frac{ds}{s}K_\mu(s)\right]\int_{z_1}^{+\infty}dz_2z_2K_{\nu'}(z_2)$$
$$\times\left[K_\mu(z_2)\int_0^{z_2}\frac{ds}{s}I_\mu(s)+I_\mu(z_2)\int_{z_2}^{\infty}\frac{ds}{s}K_\mu(s)\right], \tag{73}$$

with $\nu'=(2/\beta)\sqrt{(p+\omega)/\sigma}$. The thermodynamic limit $L\to\infty$ is obtained as

$$\lim_{L\to\infty}\hat{C}_L^{free}(p)\equiv\lim_{L\to\infty}\int_0^L dle^{-pl}C_L(l) \tag{74}$$
$$=\lim_{\omega\to 0}[\omega\hat{C}^{free}(p,\omega)]$$
$$=8\int_0^{+\infty}dz_1z_1I_\nu(z_1)K_0(z_1)\times\int_{z_1}^{+\infty}dz_2z_2K_\nu(z_2)K_0(z_2) \tag{75}$$

and thus coincides with the result (53) for correlation of two particles of the thermal packet in Sinai diffusion on the infinite line. It is shown in Eq. (C5) of Appendix (C) that periodic boundary conditions also yield the same result (75).

C. Probability distribution of the partition function

The probability distribution of the partition function,

$$\mathcal{Z}_L=\int_0^L dxe^{-\beta U(x)}, \tag{76}$$

has already been computed in Ref. [24], but to compare with our result (37) for the infinitely deep valley, we need to consider the modified partition function

$$Z_L=\int_0^L dxe^{-\beta[U(x)-U_{min}]}, \tag{77}$$

where U_{min} is the minimum of $U(x)$ for $0\leqslant x\leqslant L$. Using the notations of Eq. (23), the Laplace transform of the probability distribution $P_L(Z)$ can be expressed in terms of path integrals as

$$\hat{P}_L(q)\equiv\int_0^{+\infty}dZe^{-qZ}P_L(Z)=\int_{-\infty}^0 du_0\int_{u_0}^{+\infty}du_L\int_0^L dx_0\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{U(0)=0}^{U(x_0)=u_0+\epsilon}\mathcal{D}U(x)\exp\left[-\frac{1}{4\sigma}\int_0^L dx\left(\frac{dU}{dx}\right)^2\right.$$
$$\left.-q\int_0^L dxe^{-\beta[U(x)-u_0]}\right]\Theta_{[u_0,+\infty]}\{U(x)\}\int_{U(x_0)=u_0+\epsilon}^{U(L)=u_L}\mathcal{D}U(x)\exp\left[-\frac{1}{4\sigma}\int_0^L dx\left(\frac{dU}{dx}\right)^2\right.$$
$$\left.-q\int_0^L dxe^{-\beta[U(x)-u_0]}\right]\Theta_{[u_0,+\infty]}\{U(x)\}. \tag{78}$$

The change of functional $V(x)=U(x)-u_0$ leads to

$$\hat{P}_L(q)=\int_0^{+\infty}dv_0\int_0^{+\infty}dv_L\int_0^L dx_0\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_{V(0)=v_0}^{V(x_0)=\epsilon}\mathcal{D}U(x)\exp\left[-\frac{1}{4\sigma}\int_0^L dx\left(\frac{dV}{dx}\right)^2 -q\int_0^L dx e^{-\beta V(x)}\right]\Theta_{[0,+\infty]}\{U(x)\}\int_{V(x_0)=\epsilon}^{V(L)=v_L}\mathcal{D}V(x)\exp\left[-\frac{1}{4\sigma}\int_0^L dx\left(\frac{dV}{dx}\right)^2-q\int_0^L dx e^{-\beta V(x)}\right]\Theta_{[0,+\infty]}\{U(x)\} \quad (79)$$

$$=\sigma\int_0^{+\infty}dv_0\int_0^{+\infty}dv_L\int_0^L dx_0\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}F_{[0,+\infty]}(\epsilon,x_0|v_0)F_{[0,+\infty]}(v_L,L-x_0|\epsilon), \quad (80)$$

which yields in Laplace with respect to the length L, using Eq. (B18),

$$\int_0^{+\infty}dLe^{-\omega L}\hat{P}_L(q)$$

$$=\sigma\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{+\infty}dv\hat{F}_{[0,+\infty]}(\epsilon,\omega|v)\right]^2 \quad (81)$$

$$=\left[\frac{2}{\beta\sqrt{\sigma}I_\mu\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}\right)}\int_0^s\frac{dz}{z}I_\mu(z)\right]^2, \quad (82)$$

where μ has been defined in Eq. (62). Again, the thermodynamic limit $L\to\infty$ is obtained as

$$\hat{P}_\infty(q)=\lim_{\omega\to 0}\left(\omega\int_0^{+\infty}dLe^{-\omega L}\hat{P}_L(q)\right) \quad (83)$$

$$=\frac{1}{I_0^2\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}\right)}, \quad (84)$$

which coincides with the result (38) for the infinitely deep valley. It is of course straightforward to generalize this computation to obtain the result that the joint distribution $P_\infty(y,u)$ [Eq. (31)] also coincides with the thermodynamic limit $L\to\infty$ of the finite-size joint distribution of $(x-x_{min})$ and $(u-U_{min})$, where U_{min} is the minimum of $U(x)$ for $0\leqslant x\leqslant L$ and x_{min} is the position of this minimum.

D. Conclusion

The conclusion of this section is that the statistical properties of the thermal packet for the Sinai diffusion in the infinite time limit exactly coincide with the thermodynamic limit $L\to\infty$ of the statistical properties concerning independent particles at Boltzmann equilibrium in a sample of size L, with no dependence on the boundary conditions.

VIII. LOCALIZATION PROPERTIES OF FOKKER-PLANCK EIGENFUNCTIONS

As discussed in Sec. II, the approximation (15) becomes asymptotically exact in the limit $\Gamma\to\infty$. It is thus interesting to explore the consequences of this approximation for the eigenfunctions of the Fokker-Planck operator.

A. Recall of some exact results

To discuss the properties of eigenvalues and eigenfunctions of the Fokker-Planck equation (7), it is more convenient to use the well-known transformation into an imaginary-time Schrödinger equation via the introduction of the Green function

$$G(x,t|x_0,0)=e^{(\beta/2)(U(x)-U(x_0))}P(x,t|x_0,0), \quad (85)$$

which satisfies

$$\partial_t G(x,t|x_0,0)=-H_S G(x,t|x_0,0), \quad (86)$$

$$H_S=-T\partial_x^2+\left[\frac{1}{4T}U'(x)^2-\frac{1}{2}U''(x)\right], \quad (87)$$

with the initial condition $G(x,t|x_0,0)\to\delta(x-x_0)$. This is the standard form for the Schrödinger operator H_S associated with to a diffusion process. It can be factorized as $H_S=TQ^\dagger Q$ with $Q=\partial_x+U'(x)/(2T)$ and $Q^\dagger=-\partial_x+U'(x)/(2T)$, and has thus a real positive spectrum. We consider the case of a very large but finite system where the spectrum of energies E_n is discrete. The Fokker-Planck operator H_{FP} is non-Hermitian but has the same real positive spectrum, with right and left eigenfunctions $\Phi_n^R(x)$ and $\Phi_n^L(x)$ associated with E_n. They are related to the eigenfunctions $\psi_n(x)$ of the Schrödinger operator by

$$\Phi_n^R(x)=e^{-U(x)/(2T)}\psi_n(x), \quad (88)$$

$$\Phi_n^L(x)=e^{U(x)/(2T)}\psi_n(x). \quad (89)$$

The expansion upon Fokker-Planck eigenfunctions now reads

$$P(xt|x_00)=\sum_n \Phi_n^R(x)\Phi_n^L(x_0)e^{-E_nt}. \quad (90)$$

The ground-state $n=0$ has of course for energy $E_0=0$ and corresponds to the relaxation towards Boltzmann equilibrium so that the left and right ground state eigenvectors are simply given by

$$\Phi_0^L(x)=1/\sqrt{Z_{tot}}, \tag{91}$$

$$\Phi_0^R(x)=e^{-U(x)/T}/\sqrt{Z_{tot}}, \tag{92}$$

where $Z_{tot}=\int dx e^{-U(x)/T}$ is the normalization over the finite large system.

B. Construction of an orthonormalized set of eigenfunctions for the approximate dynamics

In this section, we consider the approximation (15) for the dynamics as defining a new dynamics denoted by tilde,

$$\tilde{P}(xt|x_0 0)\equiv\sum_{V_\Gamma}\frac{1}{Z_{V_\Gamma}}e^{-\beta U(x)}\theta_{V_\Gamma}(x)\theta_{V_\Gamma}(x_0) \tag{93}$$

and try to expand it upon a basis of eigenfunctions as in Eq. (90),

$$\tilde{P}(xt|x_0 0)=\sum_n \tilde{\Phi}_n^R(x)\tilde{\Phi}_n^L(x_0)e^{-\tilde{E}_n t}. \tag{94}$$

At time t, the states in Eq. (94) with energies $\tilde{E}_n>1/t$ are negligible in the sum, and correspond in the RSRG picture to bonds that have been already decimated. The state n will become negligible in the sum (100) at a time $t_n\sim 1/\tilde{E}_n$, and this disappearance in the sum corresponds, in the RSRG picture, to a decimation at scale $\Gamma_n=T\ln t_n=-T\ln\tilde{E}_n$. The low-lying energies $\tilde{E}_n$ are thus exactly determined by the large RG scales Γ_n at which decimations occur in the system. Of course in the real system, in the case of near degeneracies of neighboring bonds, slight shifts in these levels will occur.

To determine now the eigenfunctions, we consider what happens upon this decimation at Γ_n. We may consider that the time exponential factor associated with level n has changed from $e^{-E_n t_n^-}=1$ to $e^{-E_n t_n^+}=0$ while none of the others decaying exponentials in Eq. (94) have changed (since time scales are well separated). The difference $\tilde{P}(xt_n^-|x_0 0)-\tilde{P}(xt_n^+|x_0 0)$ is thus equal to $\tilde{\Phi}_n^R(x)\tilde{\Phi}_n^L(x_0)$ On the other hand, from the RSRG point of view, what happens is simply a decimation where two valleys V_1 and V_2 have merged into a single one V'. Thus we get using Eq. (93),

$$\tilde{\Phi}_n^R(x)\tilde{\Phi}_n^L(x_0)=e^{-U(x)/T}\times\left[\frac{\theta_{V_1}(x)\theta_{V_1}(x_0)}{Z_{V_1}}+\frac{\theta_{V_2}(x)\theta_{V_2}(x_0)}{Z_{V_2}}-\frac{[\theta_{V_1}(x)+\theta_{V_2}(x)][(\theta_{V_1}(x_0)+\theta_{V_2}(x_0)]}{Z_{V_1}+Z_{V_2}}\right]. \tag{95}$$

One sees that indeed the right-hand side has the nice property to factorize into a function of x and one of x_0 and thus one can determine unambiguously $\tilde{\Phi}_n^R(x)$ and $\tilde{\Phi}_n^L(x_0)$ by fixing the constant using $\tilde{\Phi}_n^R(x)=e^{-U(x)}\tilde{\Phi}_n^L(x)$ from Eq. (89), which leads to the eigenset for $n\geqslant 1$,

$$\tilde{\Phi}_n^L(x)=\sqrt{\frac{Z_{V_1}Z_{V_2}}{Z_{V_1}+Z_{V_2}}}\left[\frac{1}{Z_{V_1}}\theta_{V_1}(x)-\frac{1}{Z_{V_2}}\theta_{V_2}(x)\right], \tag{96}$$

$$\tilde{\Phi}_n^R(x)=e^{-U(x)/T}\tilde{\Phi}_n^L(x). \tag{97}$$

One can check on Eq. (97) that the eigenfunctions have all the correct normalization and orthogonality properties. First, one has for $n\geqslant 1$,

$$\int dx\tilde{\Phi}_n^R(x)=0. \tag{98}$$

This ensures the normalization of the probability distribution $\tilde{P}(xt|x_0 0)$ for all t and x_0 as it should,

$$\int dx\tilde{P}(xt|x_0 0)=\int dx\Phi_0^R(x)\Phi_0^L(x_0)=1. \tag{99}$$

Second, one finds the correct normalization

$$\int dx\tilde{\Phi}_n^L(x)\tilde{\Phi}_n^R(x)=\int dx e^{-U(x)/T}[\Phi_n^L(x)]^2=1. \tag{100}$$

Furthermore, one can also check that the set of wave functions exactly forms an orthonormalized set

$$\int dx\tilde{\Phi}_n^L(x)\tilde{\Phi}_m^R(x)=\delta_{n,m}. \tag{101}$$

So the disorder-dependent form (97) for the eigenfunctions have all the good properties to represent via the expansion (94) the dynamics defined in Eq. (93). One may even define the following effective Fokker-Planck operator

$$\tilde{H}_{FP}=\sum_n \tilde{E}_n|\tilde{\Phi}_n^R\rangle\langle\tilde{\Phi}_n^L| \tag{102}$$

as an approximation to H_{FP} [Eq. (7)].

C. Qualitative properties of eigenfunctions

We can now discuss the typical shape of an eigenstate and compare with the qualitative features discussed in the Appendix B of Ref. [14] for the related model of one-dimensional random-hopping Hamiltonian for fermions (except that here there is no particle-hole symmetry and there is no need to distinguish even and odd sites) :

(i) An eigenstate (97) has two peaks corresponding to the minima of valleys V_1 and V_2. The eigenstate has a significant value in finite regions of order $l_T\sim T^2/\sigma$ [Eq. (40)] around these two peaks.

(ii) These two peaks are separated by a distance of order $\Gamma^2 \sim (\ln E)^2$.

(iii) Away from one of these bumps but within the valley, i.e., for distance $r \leqslant \Gamma^2$, the decay of the wave function $\Phi_n^R(x)$ is governed by the Boltzmann factor $e^{-\beta U(r)}$ that typically behaves as $e^{-c\sqrt{r}}$. In particular, on the edges of the valleys where $r \sim \Gamma^2$, this gives an amplitude of order $e^{-c'\Gamma}$.

(iv) Beyond the involved two valleys, within our simple approximation (97) with θ functions on the edges of valleys, the eigenstate is simply zero.

So to estimate the decay of the eigenstate for distances $r \geqslant \Gamma^2$, we must take into account, as was in Appendix B of Ref. [14], the fact that the two points are typically separated by a number r/Γ^2 of valleys, and that the overlap between two neighboring valleys is not exactly zero but of order $e^{-c''\Gamma}$. And thus a perturbation theory yields that the decay for distances $r \geqslant \Gamma^2$ behaves as $e^{-c''\frac{r}{\Gamma}}$. We thus recover that the localization length in the sense of asymptotic exponential decay of the associated quantum wave function thus behaves as $\lambda(E) \sim \Gamma \sim (-T \ln E)$ whereas the typical extension of an eigenstate behaves as the typical distance between the two peaks, which behaves as $l(E) \sim \Gamma^2 \sim (-T \ln E)^2$ and directly gives the low-energy behavior of the integrated density of states $N(E) \sim 1/l(E) \sim 1/(-T \ln E)^2$ [5]. We note that the length $l(E)$ is the one that appears in the averaged Green function (85) as computed in Ref. [22] for the lattice fermion model with random hopping, and computed in Ref. [7] for the present problem.

So the disorder-dependent form (97) for the eigenfunctions, even if only approximate near the edges of the valleys, give a good insight into the properties of low-energy eigenfunctions. For a study of systematic corrections to this approximation, and further results on the statistics of wave functions far in the tails, we refer the reader to Fisher [23].

IX. PROPERTIES OF METASTABLE STATES IN SINAI DIFFUSION

In the slowly relaxing systems such as glasses, granular media or disordered spin models, the notion of metastable states is an important and ubiquitous concept. In particular, the Edwards ergodicity conjecture [17] consisting in computing dynamic quantities by taking flat averages over metastable states has given rise to a lot of recent studies [15,18–21]. However, as discussed in details in Ref. [15], truly metastable stables only exist in mean-field approximations or in the zero-temperature limit, and thus to use this concept at finite temperature in finite-dimensional systems, one needs to consider metastable states with finite lifetimes. In this context we find instructive to consider what happens in Sinai diffusion from the point of view of metastable states and to compare with the general theory of glassy systems.

A. Identification of the set of metastable states at a given time *t*

In the general theory of glassy systems, it is natural to separate the dynamics into two parts: there are "fast" degrees of freedom which rapidly reach local quasiequilibrium plus a "slow" nonequilibrium part. At a given long time t, the fast motion covers a region of phase-space which can be defined as a metastable state associated with time t [15]. To Sinai diffusion, this picture directly applies. The formula (15) leads to a very direct identification of metastable states: at time t, the metastable states (i) are given by all the renormalized valleys $V_\Gamma^{(i)}$ that exist at scale $\Gamma = T \ln t$. Moreover, the formula (15) exactly corresponds to the construction of Gaveau, Schulman, and Lesne [16] as summarized in Ref. [15], where the evolution operator $e^{-tH_{FP}}$ is replaced by a projector onto the subspace of states (i) having an eigenvalue $E_n < 1/t$,

$$e^{-tH_{FP}} \sim \sum_i |P_i\rangle\langle Q_i|, \qquad (103)$$

with the following identifications: the right eigenvectors read

$$P_i(x) = \frac{e^{-\beta U(x)}}{\int_{V_\Gamma^{(i)}} dy e^{-\beta U(x)}} \theta(x \in V_\Gamma^{(i)}) \qquad (104)$$

and thus they have indeed the properties of being positive, normalized and not zero in nonoverlapping regions of space. The left eigenvectors read

$$Q_i(x) = \theta(x \in V_\Gamma^{(i)}). \qquad (105)$$

They are thus indeed one within the support of $P_i(x)$, and zero everywhere else. Moreover, the fast degrees of freedom have indeed converged towards a local Boltzmann equilibrium within the metastable state represented by the renormalized valley (104).

B. One-time dynamical quantities as averages over the set of metastable states at a given time *t*

We now consider the question of Edwards conjecture. In Sinai diffusion, within the RSRG approach [7], all one-time quantities are effectively computed by averages over all renormalized valleys, with a measure that is not *a priori* flat, but depends on the computed quantity. In particular, for a uniform initial condition, the spatial length of a renormalized valley exactly represents the size of the basin of attraction of this renormalized valley. So the rescaled size $\lambda = \sigma l/\Gamma^2 = \sigma l/(T \ln t)^2$ of the basins of attraction of the metastable states is a random variable distributed with the probability distribution

$$D(\lambda) = P^*(\cdot) *_\lambda P^*(\cdot), \qquad (106)$$

where $P^*(\lambda)$ is the distribution of the rescaled length of bonds [7]. More explicitly, its Laplace transform reads

$$\hat{D}(p) = [\hat{P}^*(p)]^2 = \frac{1}{\cosh^2 \sqrt{p}} \qquad (107)$$

and inversion yields [25]

$$D(\lambda)=\sum_{n=-\infty}^{+\infty}\left[2\pi^2\left(n+\frac{1}{2}\right)^2\lambda-1\right]e^{-\lambda\pi^2(n+1/2)^2} \quad (108)$$

$$=\frac{2}{\sqrt{\pi}\lambda^{3/2}}\sum_{m=-\infty}^{+\infty}(-1)^{m+1}m^2e^{-m^2/\lambda}. \quad (109)$$

Since Edwards conjecture is usually based on the assumption that all the basins of attraction of the various metastable states have the same size [15], it is of course a very strong hypothesis that cannot be true in general but only for special systems with special dynamics [18,21]. In Sinai diffusion, the metastable states do not have the same size and thus Edwards conjecture cannot be satisfied in general.

However, for special quantities, Edwards conjecture can be recovered. For instance, in this paper, we have computed the probability distribution $P_\infty(y,u)$ of the thermal packet, the localization parameters $Y_k(\infty)$ [Eq. (8)] and the correlation function $C(l,\infty)$ [Eq. (9)] as flat averages over infinitely deep wells. This is because in the limit $\Gamma\to\infty$, only the lower part of the Brownian valley is important, and thus there is no dependence on the size of the valley for these quantities concerning the thermal packet.

C. Hierarchical organization of metastable states associated with different times

In the case of Sinai diffusion, one knows much more than the one-time description in terms of metastable states. Indeed, the expressions (97) for the Fokker-Planck eigenvectors contain in fact all the information on the changes in time of the set of metastable states. They have an obvious hierarchical organization: two eigenstates can be either disjoint or nested (i.e., one is included in another). And the evolution in time is described by the RSRG procedure. This is why two-time aging quantities can be computed within the RSRG method [7,26].

APPENDIX A: USEFUL PROPERTIES OF BESSEL FUNCTIONS

The Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are two linearly independent solutions of

$$z^2f''(z)+zf'(z)-(z^2+\nu^2)f(z)=0. \quad (A1)$$

Series expansion in z reads

$$I_0(z)=\sum_{k=0}^{+\infty}\frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}, \quad (A2)$$

$$K_0(z)=-I_0(z)\ln\frac{z}{2}+\sum_{k=0}^{+\infty}\frac{\psi(k+1)}{(k!)^2}\left(\frac{z}{2}\right)^{2k}, \quad (A3)$$

and

$$I_\nu(z)=\sum_{k=0}^{+\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{\nu+2k}, \quad (A4)$$

$$K_\nu(z)=\frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin\pi\nu}. \quad (A5)$$

The Wronskian property

$$I_\nu(z)K'_\nu(z)-I'_\nu(z)K_\nu(z)=-\frac{1}{z} \quad (A6)$$

gives

$$\int_z^{+\infty}ds\frac{1}{sI_0^2(s)}=-\left[\frac{K_0(s)}{I_0(s)}\right]_z^{+\infty}=\frac{K_0(z)}{I_0(z)}, \quad (A7)$$

$$\int_z^{+\infty}ds\frac{K_0(s)}{sI_0^3(s)}=-\frac{1}{2}\left[\frac{K_0^2(s)}{I_0^2(s)}\right]_z^{+\infty}=\frac{K_0^2(z)}{2I_0^2(z)}, \quad (A8)$$

$$\int_z^{+\infty}ds\frac{K_0^2(s)}{sI_0^4(s)}=-\frac{1}{3}\left[\frac{K_0^3(s)}{I_0^3(s)}\right]_z^{+\infty}=\frac{K_0^3(z)}{3I_0^3(z)}. \quad (A9)$$

Another useful integral is for any $k>0$,

$$\int_0^{\infty}dzz^{2k-1}K_0^2(z)=\frac{\sqrt{\pi}\Gamma^3(k)}{4\Gamma\left(k+\frac{1}{2}\right)}. \quad (A10)$$

Equation (A1) leads to

$$\frac{d}{ds}\left[\frac{s^2}{2}\{I_0^2(s)-[I'_0(s)]^2\}\right]=sI_0^2(s), \quad (A11)$$

$$\frac{d}{ds}\left[\frac{s^2}{2}(K_0^2(s)-[K'_0(s)]^2)\right]=sK_0^2(s), \quad (A12)$$

$$\frac{d}{ds}\left[\frac{s^2}{2}[I_0(s)K_0(s)-I'_0(s)K'_0(s)]\right]$$

$$=sI_0(s)K_0(s). \quad (A13)$$

Differentiation with respect to order

$$I_\nu(z)=I_0(z)-\nu K_0(z)+O(\nu^2), \quad (A14)$$

$$K_\nu(z)=K_0(z)+O(\nu^2). \quad (A15)$$

APPENDIX B: EXPLICIT COMPUTATIONS FOR INFINITELY DEEP VALLEYS

1. Explicit expression for $F_{[0,\Gamma]}(u,x|u_0)$

The path integral $F_{[0,\Gamma]}(u,x|u_0)$ defined in Eq. (23) can be obtained via the Feynman-Kac formula. Indeed, as a function of the variables (u,x), it satisfies the imaginary-time Schrödinger equation

$$\partial_xF=\sigma\partial_u^2F-qe^{-\beta u}F \quad (B1)$$

with the initial condition at $x=0$,

$$F(u,0)=\delta(u-u_0) \tag{B2}$$

and the absorbing boundary conditions at $u=0$ and $u=\Gamma$,

$$F(0,x)=0, \tag{B3}$$

$$F(\Gamma,x)=0. \tag{B4}$$

The Laplace transform with respect to x,

$$\hat{F}(u,p)=\int_0^{+\infty} dx e^{-px}F(u,x), \tag{B5}$$

satisfies the system

$$\sigma\partial_u^2\hat{F}-qe^{-\beta u}\hat{F}-p\hat{F}=-\delta(u-u_0), \tag{B6}$$

$$\hat{F}(0,p)=0,$$

$$\hat{F}(\Gamma,p)=0.$$

Let us introduce two linearly independent solutions $\phi_1(u,p)$ and $\phi_2(u,p)$ of the equation.

$$\sigma\partial_u^2\hat{F}-qe^{-\beta u}\hat{F}-p\hat{F}=0. \tag{B7}$$

In terms of the new variables,

$$z=\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}e^{-\beta u/2}, \tag{B8}$$

$$\nu=\frac{2}{\beta}\sqrt{\frac{p}{\sigma}}, \tag{B9}$$

the equation becomes

$$z^2\hat{F}''(z)+z\hat{F}'(z)-(z^2+\nu^2)\hat{F}(z)=0. \tag{B10}$$

Two linearly independent solutions of this equation are the Bessel functions $I_\nu(z)$ and $K_\nu(z)$, so we choose

$$\phi_1(u,p)=I_\nu(z)=I_{(2/\beta)\sqrt{p/\sigma}}\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}e^{-\beta u/2}\right), \tag{B11}$$

$$\phi_2(u,p)=K_\nu(z)=K_{(2/\beta)\sqrt{p/\sigma}}\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}e^{-\beta u/2}\right), \tag{B12}$$

and the Wronskian of these two solutions reads

$$w=\phi_1(u)\phi_2'(u)-\phi_2(u)\phi_1'(u)=\frac{\beta}{2}. \tag{B13}$$

We now introduce the function

$$E(u,v,p)=\frac{1}{w}[\phi_1(u,p)\phi_2(v,p)-\phi_2(u,p)\phi_1(v,p)] \tag{B14}$$

and express two solutions of Eq. (B7) which vanish respectively at 0 and Γ as

$$\Phi_-(u,p)=E(0,u,p), \tag{B15}$$

$$\Phi_+(u,p)=E(u,\Gamma,p). \tag{B16}$$

Their Wronskian reads

$$W(p)=\Phi_-'(u,p)\Phi_+(u,p)-\Phi_-(u,p)\Phi_+'(u,p)$$
$$=E(0,\Gamma,p). \tag{B17}$$

The solution of the system (B6) now reads

$$\hat{F}_{[0,\Gamma]}(u,p|u_0)=\frac{\Phi_-(\min(u,u_0),p)\Phi_+(\max(u,u_0),p)}{\sigma W(p)}$$
$$=\frac{E(0,\min(u,u_0),p)E(\max(u,u_0),\Gamma,p)}{\sigma E(0,\Gamma,p)}. \tag{B18}$$

2. Explicit expression for $R_\infty(q)$

Specializing Eq. (B18) to the case $u_0=\epsilon$ and $u=\Gamma-\epsilon$ and expanding

$$E(0,\epsilon,p)=\epsilon+O(\epsilon^2), \tag{B19}$$

$$E(\Gamma-\epsilon,\Gamma,p)=\epsilon+O(\epsilon^2), \tag{B20}$$

we get

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\hat{F}_{[0,\Gamma]}(\Gamma-\epsilon,p|\epsilon)=\frac{1}{\sigma E(0,\Gamma,p)}. \tag{B21}$$

Then Eq. (24) yields

$$R_\Gamma(q)=\mathcal{N}(\Gamma)\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\hat{F}_{[0,\Gamma]}(\Gamma-\epsilon,p=0|\epsilon) \tag{B22}$$

$$=\frac{\mathcal{N}(\Gamma)}{\sigma E(0,\Gamma,p=0)}. \tag{B23}$$

Using the expansions (A3) of Bessel functions, the normalization condition $R_\Gamma(q\to 0)=1$ determines the normalization

$$\mathcal{N}(\Gamma)=\sigma\Gamma, \tag{B24}$$

so that the final result reads

$$R_\Gamma(q)=\frac{\frac{\beta\Gamma}{2}}{I_0(s)K_0(se^{-\beta\Gamma/2})-K_0(s)I_0(se^{-\beta\Gamma/2})} \tag{B25}$$

with $s=(2/\beta)\sqrt{q/\sigma}$.

3. Explicit expression for $S_\Gamma(y,u,q)$

We now turn to the evaluation of Eq. (26) in Laplace with respect to y

$$\hat{S}_\Gamma(p,u,q) \equiv \int_0^{+\infty} dy e^{-py} S_\Gamma(y,u,q) \tag{B26}$$

$$= \mathcal{N}(\Gamma) e^{-\beta u} \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^2} \hat{F}_{[0,\Gamma]}(\Gamma-\epsilon,0|u) \hat{F}_{[0,\Gamma]}(u,p|\epsilon)\right). \tag{B27}$$

Using

$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\hat{F}_{[0,\Gamma]}(u,p|\epsilon)\right) = \frac{E(u,\Gamma,p)}{\sigma E(0,\Gamma,p)} \tag{B28}$$

and

$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\hat{F}_{[0,\Gamma]}(\Gamma-\epsilon,0|u)\right) = \frac{E(0,u,0)}{\sigma E(0,\Gamma,0)}, \tag{B29}$$

we get

$$\hat{S}_\Gamma(p,u,q) = \frac{\Gamma}{\sigma E(0,\Gamma,0)} e^{-\beta u} E(0,u,0) \frac{E(u,\Gamma,p)}{E(0,\Gamma,p)}. \tag{B30}$$

In the limit $\Gamma \to \infty$, using the expansions (A5) we get

$$\lim_{\Gamma\to\infty}\left(\frac{E(u,\Gamma,p)}{E(0,\Gamma,p)}\right) = \frac{I_\nu((2/\beta)\sqrt{q/\sigma} e^{-\beta u/2})}{I_\nu((2/\beta)\sqrt{q/\sigma})} \tag{B31}$$

and

$$\lim_{\Gamma\to\infty}\left(\frac{\Gamma}{E(0,\Gamma,0)}\right) = \frac{1}{I_0((2/\beta)\sqrt{q/\sigma})}, \tag{B32}$$

so that we obtain the formula (27) given in the text.

APPENDIX C: COMPUTATION OF EQUILIBRIUM FUNCTIONS IN LARGE SYSTEMS

1. Basic path integral

The Laplace transform with respect to l of the basic path integral (59) satisfies the same equation as Eq. (B6) but the boundary conditions are now at $u \to \pm\infty$. So using again the notations (B9) the solution reads [Eq. (B18)]

$$\hat{G}(u,p|u_0) = \frac{2}{\beta\sigma} \times K_\nu\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}} e^{-(\beta/2)\min(u,u_0)}\right) \times I_\nu\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}} e^{-(\beta/2)\max(u,u_0)}\right). \tag{C1}$$

2. Two-point correlation $C^{eq}(l)$ for periodic boundary conditions

For periodic boundary conditions, the two-point correlation may be expressed in terms of the path integral (59) as

$$C_L^{periodic}(l) = 2\sqrt{4\pi\sigma L}\int_0^{L-l} dx \int_0^{+\infty} dq q \times \int_{-\infty}^{+\infty} du_1 e^{-\beta u_1} \int_{-\infty}^{+\infty} du_2 e^{-\beta u_2} \times G(0,L-x-l|u_2) G(u_2,l|u_1) G(u_1,x|0), \tag{C2}$$

which yields in Laplace transform

$$\hat{c}(p,\omega) \equiv \int_0^{+\infty} dL e^{-\omega L} \int_0^L dl e^{-pl} \left(\frac{C_L^{free}(l)}{\sqrt{L}}\right) \tag{C3}$$

$$= \frac{16\sqrt{4\pi\sigma}}{\beta\sigma} \int_0^{+\infty} \frac{ds}{s} \int_0^{+\infty} dz_1 z_1 I_{\nu'}(z_1) \times [K_\mu(z_1) I_\mu(s)\theta(z_1-s) + I_\mu(z_1) K_\mu(s)\theta(s-z_1)] \times \int_{z_1}^{+\infty} dz_2 z_2 K_{\nu'}(z_2) \times [K_\mu(z_2) I_\mu(s)\theta(z_2-s) + I_\mu(z_2) K_\mu(s)\theta(s-z_2)] \tag{C4}$$

with $\mu = (2/\beta)\sqrt{\omega/\sigma}$ and $\nu' = (2/\beta)\sqrt{(p+\omega)/\sigma}$. The thermodynamic limit $L \to \infty$ is obtained as

$$\hat{C}_\infty^{periodic}(p) = \lim_{\omega\to 0}\left[\frac{\sqrt{\omega}}{\sqrt{\pi}}\hat{c}(p,\omega)\right] = 8\int_0^{+\infty} dz_1 z_1 I_\nu(z_1) K_0(z_1) \times \int_{z_1}^{+\infty} dz_2 z_2 K_\nu(z_2) K_0(z_2). \tag{C5}$$

APPENDIX D: RELATION WITH GOLOSOV THEOREM

As explained in the Introduction, the theorem of Golosov [2] states that $P_\infty(y)$ is equal to $P_G(y)$ given by Eq. (3). The purpose of this appendix is to show this formula of Golosov gives the same result as Eq. (33) obtained in the text. We first rewrite Eq. (3) as

$$P_G(y) = \lim_{L\to\infty}\left(\int_0^\infty dq \left\langle\left\langle \exp\left(-q\int_0^L dt e^{-\beta\rho(t)}\right)\right\rangle\right\rangle_{\{\rho\}} \right. \tag{D1}$$

$$\left. \left\langle\left\langle \exp\left[-\beta r(y) - q\int_0^L dt e^{-\beta r(t)}\right]\right\rangle\right\rangle_{\{r\}}\right). \tag{D2}$$

Since the Bessel processes $\{r(t)\}$ and $\{\rho(t)\}$ may be considered as radial parts of free three-dimensional Brownian motion, the Feynman-Kac formula yields

$$\left\langle\left\langle \exp\left(-q\int_0^L dt e^{-\beta\rho(t)}\right)\right\rangle\right\rangle_{\{\rho\}} = \int_0^{+\infty} 4\pi R_L^2 dR_L G_p(R_L,L|0), \tag{D3}$$

$$\left\langle\left\langle \exp\left[-\beta r(y) - q\int_0^L dt e^{-\beta r(t)}\right]\right\rangle\right\rangle_{\{r\}}$$

$$= \int_0^{+\infty} 4\pi R_L^2 dR_L \int_0^{+\infty} 4\pi R^2 dR$$

$$\times G_q(R_L, L-y|R) e^{-\beta R} G_q(R,y|0), \qquad \text{(D4)}$$

where $G_q(R,l|0)$ satisfies the imaginary-time Schrödinger equation

$$\partial_l G_q(R,l|R_0) = -H_q G_q(R,l|R_0), \qquad \text{(D5)}$$

where

$$H_q = -\frac{\sigma}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial f}{\partial R}\right) + q e^{-\beta R} \qquad \text{(D6)}$$

is the radial restriction of the corresponding three-dimensional Hamiltonian. We have, moreover, the initial condition

$$G_q(R, l \to 0|R_0) \to \frac{1}{4\pi R^2}\delta(R-R_0). \qquad \text{(D7)}$$

The identity

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) = \frac{1}{R}\frac{\partial^2}{\partial R^2}(Rf)$$

leads to the change of function

$$g_q(R,l|R_0) = 4\pi R R_0 G_q(R,l|R_0). \qquad \text{(D8)}$$

This new function g_q satisfies the one-dimensional Schrödinger equation

$$\partial_l g_q(R,l|R_0) = -h_q g_q(R,l|R_0), \qquad \text{(D9)}$$

$$h_q = -\sigma \frac{d^2}{dR^2} + q e^{-\beta R}, \qquad \text{(D10)}$$

on the semi-infinite line $R \geqslant 0$ with the absorbing boundary condition at $R=0$,

$$g_q(R=0,l|R_0) = 0 \qquad \text{(D11)}$$

and with the initial condition

$$g_q(R, l\to 0|R_0) \to \delta(R-R_0). \qquad \text{(D12)}$$

By comparison with Eqs. (B1),(B2),(B4), we immediately obtain

$$g_q(R,l|R_0) = \lim_{\Gamma\to\infty} F_{[0,\Gamma]}(R,l|R_0). \qquad \text{(D13)}$$

Since the function $G_q(R,l|0)$ has to be obtained as the limit

$$G_q(R,l|0) = \frac{1}{R}\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon} g_q(R,l|\epsilon)\right), \qquad \text{(D14)}$$

we have, using Eqs. (B9) and (A5),

$$\langle e^{-q\int_0^\infty dt e^{-\rho(t)}}\rangle_{\{\rho\}}$$

$$= \lim_{L\to\infty}\int_0^\infty dR R \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon} g_q(R,L|\epsilon)\right) \qquad \text{(D15)}$$

$$= \lim_{\nu\to 0}\left[\frac{4/(\beta\nu)^2}{I_\nu\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}\right)}\int_0^\infty dR R I_\nu\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}} e^{-\beta(R/2)}\right)\right] \qquad \text{(D16)}$$

$$= \frac{1}{I_0\left(\frac{2}{\beta}\sqrt{\frac{q}{\sigma}}\right)}, \qquad \text{(D17)}$$

in agreement with Eq. (25).

Similarly, we compute the Laplace transform with respect to y

$$\int_0^{+\infty} dy e^{-py}\left\langle e^{-\beta r(y)} \exp\left(-q\int_0^{+\infty} dt e^{-\beta r(t)}\right)\right\rangle_{\{r\}} \qquad \text{(D18)}$$

$$= \lim_{s\to 0}\left[s\int_0^\infty dR R\int_0^\infty dR' \hat{g}_q(R,s|R') e^{-\beta R'} \qquad \text{(D19)}\right.$$

$$\left.\times \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\hat{g}_q(R',p|\epsilon)\right)\right], \qquad \text{(D20)}$$

which gives the same result as Eq. (27).

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